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Universal *R* Matrix of Two-Parameter Deformed Quantum Group $U_{qs}(SU(2))$

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A method of direct derivation for the universal *R* matrix of the two-parameter deformed quantum group $U_{qs}(SU(2))$ is given. It is simple and efficient for quantum groups of low rank at least.

1. INTRODUCTION

Quantum groups arise in the study of quantum inverse scattering methods (Faddeev, 1981) and there have been many extensive studies on their representation theory and structure (for example, Macfarlane, 1989; Biedenharn, 1989; Sum and Fu, 1989; Chakrabarti and Jagannathan, 1991; Jing and Cuypers, 1993). In this paper, we will derive the universal *R* matrix of the two-parameter deformed quantum group $U_{qs}(SU(2))$ by a method of direct derivation which is different from that of the quantum double.

2. THE UNIVERSAL *R* MATRIX OF THE TWO-PARAMETER DEFORMED QUANTUM GROUP $U_{qs}(SU(2))$

The generators of the two-parameter deformed quantum group $U_{qs}(SU(2))$ satisfy the commutative relations (Jing and Cuypers, 1993)

$$[K_0, K_\pm] = \pm K_\pm, \qquad s^{-1}K_+K_- - sK_-K_+ = s^{-2K_0}[2K_0] \tag{1}$$

where we have used the notation $[x] = (q^x - q^{-x})/(q - q^{-1})$. For future convenience, we also define the two-parameter deformation brackets

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$$[x]_{qs} = s^{1-x}[x], \qquad [x]_{qs^{-1}} = s^{x-1}[x] \tag{2}$$

The two-parameter deformed quantum group $U_{qs}(SU(2))$ is a Hopf algebra. Its coproduct takes the form (Yu *et al.*, 1995)

$$\Delta(K_0) = K_0 \otimes 1 + 1 \otimes K_0 \tag{3}$$

$$\Delta(K_\pm) = K_\pm \otimes (s^{-1}q)^{K_0} + (sq)^{-K_0} \otimes K_\pm \tag{4}$$

Letting the symbol $\overline{\Delta}$ denote the inverse coproduct of Δ, i.e.,

$$\overline{\Delta} = T\Delta \tag{5}$$

where T stands for the twisted mapping, one has

$$T(x \otimes y) = y \otimes x, \qquad \forall x, y \in U_{qs}(SU(2)) \tag{6}$$

One has the following relation between Δ and $\overline{\Delta}$:

$$\overline{\Delta}(a)R = R\Delta(a), \qquad a \in U_{qs}(SU(2)) \tag{7}$$

with R the universal matrix; R can be represented by

$$R = \sum_i a_i \otimes b_i \tag{8}$$

Equation (8) satisfies the Yang–Baxter equation (Yang, 1967; Baxter, 1972)

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{9}$$

where

$$R_{12} = R \otimes 1, \qquad R_{13} = \sum_i a_i \otimes 1 \otimes b_i, \qquad R_{23} = 1 \otimes R$$

Utilizing the Baker–Campbell–Hausdorff formula

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \cdots \tag{10}$$

we have

$$q^{2K_0}K_-^m = q^{-2m}K_-^m q^{2K_0}, \qquad q^{-2K_0}K_-^m = q^{2m}K_-^m q^{-2K_0} \tag{11}$$

$$q^{2K_0}K_+^m = q^{2m}K_+^m q^{2K_0}, \qquad q^{-2K_0}K_+^m = q^{-2m}K_+^m q^{-2K_0} \tag{12}$$

By virtue of Eqs. (11) and (12), we can prove the following relations:

$$s^k K_-^k K_+ = s^{-k}K_+K_-^k - K_-^{k-1}s^{-2K_0}[k]_{qs^{-1}}[2K_0 - (k-1)] \tag{13}$$

$$s^k K_- K_+^k = s^{-k}K_+^k K_- - K_+^{k-1}s^{-2K_0}[k]_{qs}[2K_0 + (k-1)] \tag{14}$$

For simplicity, let x and x' stand for the first and the second operator in the tensor product $U_{qs}(SU(2)) \otimes U_{qs}(SU(2))$, respectively, such that Eqs. (3) and (4) can be rewritten as

$$\Delta(K_0) = K_0 + K_0' \tag{15}$$

$$\Delta(K_\pm) = K_\pm(s^{-1}q)^{K_0'} + (sq)^{-K_0}K_\pm' \tag{16}$$

Equations (7) becomes

$$\overline{\Delta}(K_0)R(x, x') = R(x, x')\Delta(K_0) \tag{17}$$

$$\overline{\Delta}(K_\pm)R(x, x') = R(x, x')\Delta(K_\pm) \tag{18}$$

In order to get an explicit formula from $R(x, x')$, we assume

$$R(x, x') = \sum_{l=0}^{\infty} C_l(K_0, K_0')K_-^l(K_+')^l \tag{19}$$

where $C_l(K_0, K_0')$ denotes a functional of operators K_0 and K_0' as well as parameters l, q, and s. It is easy to check that Eq. (19) satisfies Eq. (17) trivially. Substituting Eq. (19) into Eq. (18) and utilizing Eqs. (1), (11), (12), (15), and (16), we get four equations

$$s^{-K_0'+l}q^{-K_0'-l}C_l(K_0 - l, K_0' + l) = (s^{-1}q)^{K_0'}C_l(K_0 - l + 1, K_0' + l) \tag{20}$$

$$\begin{aligned}(s^{-1}q)^{K_0-l}&C_l(K_0 - l, K_0' + l)\\ &+ s^{-2K_0-K_0'}q^{-K_0'-l-1}[l + 1]_{qs^{-1}}[2K_0 - l]C_{l+1}(K_0 - l + 1, K_0' + l + 1)\\ &= (sq)^{-K_0}C_l(K_0 - l, K_0' + l + 1)\end{aligned} \tag{21}$$

$$s^{-K_0-l}q^{K_0-l}C_l(K_0 - l, K_0' + l) = (sq)^{-K_0}C_l(K_0 - l, K_0' + l - 1) \tag{22}$$

$$\begin{aligned}(sq)^{-K_0'-l}&C_l(K_0 - l, K_0' + l) - (s^{-1}q)^{K_0'}C_l(K_0 - l - 1, K_0' + l)\\ &= s^{-2K_0'-K_0}q^{K_0-l-1}[l + 1]_{qs}[2K_0' + l]C_{l+1}(K_0 - l - 1, K_0' + l + 1)\end{aligned} \tag{23}$$

Assuming that

$$C_l(K_0, K_0') = \tilde{C}_l s^{aK_0K_0'+bK_0+cK_0'}q^{dK_0K_0'+eK_0+fK_0'} \tag{24}$$

and substituting Eq. (24) into Eqs. (20) and (22), one has

$$a = 0, \qquad b = c = l, \qquad d = -2, \qquad e = l, \qquad f = -l \tag{25}$$

so $C_l(K_0, K_0')$ takes the form

$$C_l(K_0, K_0') = \tilde{C}_l s^{l(K_0+K_0')}q^{-2K_0K_0'+l(K_0-K_0')} \tag{26}$$

Putting Eq. (26) into Eqs. (22) and (24), one gets

$$\tilde{C}_{l+1} = \tilde{C}_l(1 - q^2)q^l/[l + 1] \tag{27}$$

The recurrence formula is

$$\tilde{C}_l = \tilde{C}_0(1 - q^2)^l q^{l(l-1)/2}/[l]! \tag{28}$$

where $[l]! = [l][l-1]\ldots[2][1]$, and $\tilde{C}_0 = \text{const}$. (In the following discussion we let $\tilde{C}_0 = 1$.) Then $R(x, x')$ can be expressed as

$$R(x, x') = \sum_{l=0}^{\infty} \frac{(1 - q^2)^l q^{l(l-1)/2}}{[l]!} s^{l(K_0+K_0')} q^{-2K_0K_0'+l(K_0-K_0')} K_-^l K_+'^l \tag{29}$$

We now check whether Eq. (29) holds for the Yang–Baxter equation, i.e.,

$$R(x, x')R(x, x'')R(x', x'') = R(x', x'')R(x, x'')R(x, x') \tag{30}$$

The left-hand of Eq. (30) is equal to

$$\begin{aligned} &R(x, x')R(x, x'')R(x', x'') \\ &= \sum_{M,N=0}^{\infty} q^{-2K_0K_0'-2K_0K_0''-2K_0'K_0''} q^{M(K_0-K_0')+N(K_0'-K_0'')-NM} \\ &\quad\times s^{M(K_0+K_0')+N(K_0'+K_0'')} K_-^M K_+''^N \sum_{l=0}^{\min(M,N)} \tilde{C}_{M-l}\tilde{C}_l\tilde{C}_{N-l} q^{l(2K_0'+2N-l)} \\ &\quad\times s^{-l(2K_0'-2M+l)-MN} K_+'^{M-l} K_-'^{N-l} \end{aligned} \tag{31}$$

and for the right-hand of Eq. (30), we get

$$\begin{aligned} &R(x', x'')R(x, x'')R(x, x') \\ &= \sum_{M,N=0}^{\infty} q^{-2K_0K_0'-2K_0K_0''-2K_0'K_0''} q^{M(K_0-K_0')+N(K_0'-K_0'')-MN} \\ &\quad\times s^{M(K_0+K_0')+N(K_0'+K_0'')} K_-^M K_+''^N \sum_{l=0}^{\min(M,N)} \tilde{C}_{N-l}\tilde{C}_l\tilde{C}_{M-l} q^{-l(2K_0'-2M+l)} \\ &\quad\times s^{-l(2K_0'+2N-l)+MN} K_-'^{N-l} K_+'^{M-l} \end{aligned} \tag{32}$$

On the other hand, for all nonnegative integers M and N, the following relation holds:

$$\sum_{l=0}^{\min(M,N)} \tilde{C}_{M-l}\tilde{C}_l\tilde{C}_{N-l}q^{l(2K_0'+2N-l)}s^{-l(2K_0'-2M+l)-MN}K_+'^{M-l}K_-'^{N-l}$$

$$= \sum_{l=0}^{\min(M,N)} \tilde{C}_{N-l}\tilde{C}_l\tilde{C}_{M-l}q^{-l(2K_0'-2M+l)}s^{-l(2K_0'+2N-l)+MN}K_-'^{N-l}K_+'^{M-l} \quad (33)$$

3. CONCLUSION

From Eqs. (31)–(33), we conclude that Eq. (29) is the universal R matrix of the two-parameter deformed quantum group $U_{qs}(SU(2))$. The method used above is simple and efficient for the quantum group of low rank at least.

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